THE MATHEMATICS OF ARYABHATA.

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Introductory.

India has always been more a land of philosophy and metaphysics than a land of materialism and scientific research. Unlike Greece and Arabia, ancient India could boast of few persons devoted to the advancement of mathematics as a science by itself. There are very few classical Indian books dealing only with pure mathematics, while almost every Indian astronomical work contains incidentally some chapters in mathematics, giving briefly the lemmas useful for subsequent astronomical calculations.

Mathematics in India was brought up mainly as a handmaid of astronomy, which was itself but an auxiliary to the study of the Vedas and the performance of daily rituals and sacrifices enjoined in the Vedas to please the gods. This accounts for the fact that Indian mathematics is essentially practical and does not contain several water-tight compartments such as Geometry, Algebra, Arithmetic and Trigonometry. There is no such elaborate theory as in Greek mathematics while some theory that is occasionally given takes a practical form. The fragmentary and apparently incoherent presentation of mathematical ideas in the classical Indian treatises has led some of the modern oriental scholars of the type of Mr. G. R. Kaye* to suspect Indian originality and to indulge in pleasant and fanciful hopes that the Indian works record the mathematics of Hypatia or the contents of the lost books of Diophantus or even those of early Chinese works. To quote one instance, there is an attempt to trace to Chinese sources, the origin of the use of the names of colours for variables in Indian Algebra.†

Though ancient India had always given a sort of marginal attention to the study of mathematics, yet the peculiar Indian genius with its marvellous gifts of intuition was destined to give to the rest of the world (though there are oriental scholars who will entirely deny this claim) the important basic

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* *Vide* 'The Sources of Hindu Mathematics', by G. R. Kaye (J. R. A. S., 1910) and 'Indian Mathematics' (p. 15), by G. R. Kaye.

† Colour is the most common concrete symbol for distinguishing things. In accounts, entries in black and red inks have different kinds of significance: in a world-map, countries belonging to different nations are coloured differently. It is a universal practice to adopt the colour principle to point out differences. No wonder, therefore, that the ancient Hindus should have naturally thought of the names of the different colours for denoting different variables. I believe that our ancestors were not really so *colour-blind* as to be compelled to borrow the colour-idea from the Chinese.

ideas in mathematics—the place-value system of notation in Arithmetic, the generalizations of Algebra, the sine-function in Trigonometry and the foundations of Indeterminate Analysis. A nation that could compress all its Grammar, all its Philosophy, into a few Sūtras—a unique feature of Indian literature—need not go a-borrowing for symbols to express its mathematical ideas. The ancient Hindus had a special genius for algebraic symbolism. Symbols were their speciality. Hence they were eminently fitted to lay the foundations of mathematics, which they did admirably. Indeed, as H. T. Colebrooke has remarked, had an earlier translation of the Hindu mathematical treatises been made and given to the public, especially to the early mathematician in Europe, the progress of mathematics would have been much more rapid, since algebraic symbolism would have reached its perfection long before the days of Descartes, Pascal and Newton.

The Indian mathematical works are, as a rule, written in verse and the poetic license adds to the obscurity of the language. Besides, they are very brief containing merely rules, results, and sometimes a number of problems with solution, but very rarely a fully worked out mathematical argument. It is just in keeping with the Indian tradition to make the text as brief and concise as possible, so that the whole of it may be easily learned by heart and remembered, the explanations being left to be learnt orally from the Gurus or teachers.

Aryabhata—His Age and Works.

In the whole range of mathematical and astronomical literature of ancient India, one of the most prominent and scientific writers is Āryabhaṭa of Kusumapura born in the year 3577 Kali, corresponding to 476 A.D. He himself says :

षष्ट्यब्दानां षष्टिर्यदा व्यतीतास्त्रयश्च युगपादाः ।
त्र्यधिका विंशतिरब्दास्तदेह मम जन्मनो व्यतीताः ॥

in the section कालक्रियापाद of his work styled Laghu-Āryabhaṭīyam or Āryabhaṭa-tantra. Mr. G. R. Kaye, in his article Āryabhaṭa, J. A. S. B., IV, 17 (1908) says that Brahmagupta nowhere in his mathematical sections mentions Āryabhaṭa nor does Bhāskara. But Bhāskara has referred to Āryabhaṭa in the following terms : 'अत एव सूर्यसिद्धान्तार्यभटतन्त्रेष्वेतान्येव' in connection with his sine-tables, which are identically the same as Āryabhaṭa's, except in one place.

As regards the identity of Āryabhaṭa, there is an element of doubt. There is another Āryabhaṭa who is known by his work, Mahāsiddhānta ; he refers to the old Āryabhaṭa thus :

एवं परोपकृतये स्वोक्तयोक्तं खेचरानयनं ।
किंचित्पूर्वागमसममुक्तं विप्राः पठन्त्विदं नान्ये ॥

वृद्धार्यभटप्रोक्तात् सिद्धान्तायन्महाकालात् ।
पाठैर्गतमुच्छेदं विशेषितं तन्मया स्वोक्तया ॥

in the fourteenth sloka of Pātādhikāra. But Alberuni calls Āryabhaṭa the younger, "that one from Kusumapura". Again the confusion is worse confounded by Sudhākara Dvivedi in his Preface to the Mahāsiddhānta. He gives dubious* references from Bhāskara to show that Bhāskara did not know of the older Āryabhaṭa but only the younger one. I believe that the discrepancies in the references to Āryabhaṭa of Kusumapura must be due to incomplete and erroneous manuscripts of Āryabhaṭīyam being in circulation.

In this connection, it is worthy of note that Āryabhaṭa's treatise on Algebra has been translated into Latin by one G. de-Lunis, an Italian mathematician of the thirteenth century and there is a manuscript copy of the translation in the Bibliotheque Nationale de Florence (*vide* L. Inter. des. Maths., July and August, 1909). Probably a reference to this Latin translation may clear some of the doubts regarding Āryabhaṭa's identity. In 1874 Dr. Kern brought out the first edition of the text Āryabhaṭīyam with the long commentary of Paramādiśvara and in 1879, Rodet gave a French translation of the Gaṇitapāda, the mathematical portion of the text with very valuable and interesting notes, while Thibaut in 1899 gave a summary of the literature about Āryabhaṭa and G. R. Kaye in 1908 (J. A. S. B., Vol. IV, 17) published his notes on Āryabhaṭa with a literal English translation and commentary of the text.

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* *Vide* pp. 22, 23, Mahāsiddhānta, edited with his own commentary by Mahāmahōpādhyāya Sudhākara Dvivedi :

"Bhāskara says in his Vāsanābhāshya of sloka 52 of Bhuvanakōsa of Golādhyāya 'अतोऽयुतद्वयव्यासे द्विकाग्न्यष्टयमर्त्तुमितः परिधिरार्यभटाद्यैरङ्गीकृतः ।........' This rule is found in Laghu-Āryabhaṭīyam........I think by 'आर्यभटाद्यै:' Bhaskara means many mathematicians as वृद्धार्यभट.

By Bhāskara's wording in Vāsanābhāshya of slokas 58-61, Bhuvanakōsa, Golādhyāya "यत् पुनः क्षेत्रफलमूलेन क्षेत्रफलंगुणितं घनफलं स्यादिति । तत् प्रायश्चतुर्वेदाचार्यः परमतमुपन्यस्तवान् ॥", it is clear, by *paramata*, that Bhāskara has not seen the work of Āryabhaṭa (लघ्वार्यभटीय)......

(The above statement is contradictory to that in the previous para.)

Āryabhaṭa's rule runs thus :

समपरिणाहस्यार्धं विष्कम्भार्धहतमेव वृत्तफलम् ।
तन्निजमूलेनहतं घनगोळफलं निरवशेषम् ॥

Bhāskara in his Vāsanābhāshya of sloka 65 of Grahaganita, Spashṭādhikāra says : 'अतएवार्यभटादिभिः सूक्ष्मत्वार्थं दृक्काणोदयाः पठिता :.' There is no दृक्काणोदय in Laghu-Āryabhaṭīyam but in Mahāsiddhānta, the author has mentioned दृक्काणोदय. Therefore this Āryabhaṭa (referred to must be the younger Āryabhaṭa) the author of Mahāsiddhānta......"

(The statements within the brackets are due to the present writer.)

Many works have been attributed to Āryabhaṭa, but the Āryabhaṭīyam is the only work which can be indubitably called his. It consists of four parts: the Daśagītikā Sūtra, Gaṇita, Kālakriyā and Gōla dealing respectively with astronomical tables, mathematics, the measure of time and the spherics.

Aryabhata—the Innovator in Astronomy and Father of Indian Mathematics.

As Dr. Thibaut admits elsewhere, Āryabhaṭa was the first or one of the first to expound the principles of the Indian astronomical system in a highly condensed and technical form and was original, at least so far as India was concerned, in maintaining the daily rotation of the earth on its axis. He says in his Gōlapāda, stanza 9,

अनुलोमगतिर्नौस्थः पश्यत्यचलं विलोमगं यद्वत् ।
अचलानि भानि तद्वत् समपश्चिमगानि लङ्कायाम् ॥

i.e., As one sailing forward in a boat sees the stationary objects on the bank move in the opposite direction, even so do the fixed stars appear to move due west to an observer stationed in Lanka.

But poor Āryabhaṭa could not boldly assert and maintain the above doctrine in the teeth of the orthodox popular doctrine and so he adds immediately as an alternative the popular geocentric theory also. In two other places again, Aryabhaṭa goes against the prevailing orthodox notions: in his theory of the eclipses and in his sub-division of the Chatur-yuga into four equal parts. Thus it is clear that Āryabhaṭa was an innovator in astronomy and that he attempted to reform some of the prevailing corrupt notions and doctrines, thereby incurring the displeasure of the orthodox teachers who regarded him as a heretic.

Coming to the mathematical portion of his work, which is contained partly in his Daśagītikā or ten verses and in his Gaṇitapāda of thirty-three verses, one cannot fail to note Āryabhaṭa's high originality. It cannot be denied that he is the father of Indian mathematics; for we see the later mathematical writings, *viz.*, those of Brahmagupta, Bhāskara, Mahāvīrāchārya and Śrīdhara, bear such a close similarity to Āryabhaṭa's work, barring, of course, variations in details. The subject-matter of later Indian mathematics remains practically the same as Āryabhaṭa's with the exception of two topics, *i.e.*, प्रस्तारयोगभेद or permutations and combinations and चक्रवाल or the cyclic method in solving Indeterminate Equations of the second degree. The ordinary rules of mensuration of triangles, quadrilaterals, and circles as well as the rules for finding the square-root, the cube-root, etc., agree in all the Hindu mathematical treatises as we shall see presently.

The Alphabetic Notation, Involution and Evolution.

In his Daśagītikā, Āryabhaṭa gives a peculiar notation for expressing numbers in terms of the letters of the alphabet—consonants and vowels. The twenty-five varga letters from क to म are made to represent the numbers from one to twenty-five respectively in the square or odd places, *i.e.*, in the units, hundreds, ten-thousands, etc., places and the avarga letters from य to ह representing the numbers 30, 40, . . . up to 100 are meant to occupy the even or non-square places. The nine vowels अ, इ, उ, ऋ etc., to औ (आ, ई, ऊ, ॠ having the same significance as the corresponding short vowels) attached to, or united with any consonant indicate that the value of the consonant is multiplied by 1, 100, 100^2, . . . 100^8, respectively. In conjunct consonants, the vowel attached should be considered as indicating the same multiplier for all the constituent consonants. Thus ख्युघृ = 4,320,000, ढङ्विघ्व = 1,46,564, ङिशिबुणॢष्खृ = 1,582,237,500.

But it must be noted that this system was used merely for mnemonic purposes and not followed in the Gaṇitapāda. In the second sloka, Ārya-bhaṭa gives the names of the successive powers of 10 up to 10^9.

Observing closely the notation of Āryabhaṭa, one finds in it the germ of the later place-value system; for, very often in the Daśagītikā the vowels अ, इ, उ, etc., occupy more or less the same places from right to left in a number-word as in the modern place-value notation. Thus बुफिनच=2,32,226, ञुष्खिध= 4,88,219. In all probability the positions occupied by the vowels, *i.e.*, अ in the extreme right, इ, उ, etc. each in succession in its appropriate place to the left (sometimes to the right also as in जषबिखुछृ=70,22,388) of the preceding vowel in the alphabetic sequence, mark an earlier stage in the evolution of the place-value notation.

Āryabhaṭa's notation and numeration indicate that the Hindus of that age were acquainted in a way with the principle of the position system in the Decimal or the Centesimal scale* (more probably the latter from which the former must have been a later reduction). This is specially noteworthy at a time when the Greeks were adopting the cumbrous rhetorical notation (*vide* Heath's Diophantus, Second Edition, p. 49).

After numeration and notation, Āryabhaṭa proceeds to define the square and the cube of a number and gives rules for finding the square-root and the cube-root. L. Rodet in his 'Leçons de Calcul d' Āryabhaṭa' infers from these rules that the Hindus must have had a knowledge of our modern system of

* In this connection, it may be noted that the 100-scale is employed in Taittiriya Upanishad, II Valli, 8th Anuvaka, for the description of the different orders of happiness or bliss. The bliss of Brahman is reckoned as 100^{10} times the measure of one human bliss.

(*Sacred Books of the East*, Edited by Max Müller, Vol. XV, pp. 59-61.)

arithmetical notation. But Mr. G. R. Kaye in his usual strain denies such knowledge by saying that the rule is perfectly general and applies to all notations. If the Hindus had no such notation, there would be no necessity for the numeration एक दश शत सहस्रायुत...etc. It is rather curious to observe that Mahāvīrāchārya in South India and Śrīdhara as well as Bhāskara and Brahmagupta give more or less identical rules for the extraction of the square- and the cube-root, while no method of extracting the cube-root is given by any early Greek writer. (*Vide* 'Greek Mathematics' by T. L. Heath, Vol. I, p. 63 and Vol. II, p. 341.)

Some Mensuration Formulae.

Next, the author proceeds to give some mensuration formulæ, some of which are obviously wrong, probably due to wrong and careless generalization from analogy. Thus the area of a triangle is given to be equal to the product of half the base and the corresponding altitude but the volume of a solid with six edges, being considered as the analogue of the triangle in three dimensions, is given to be also equal to the product of half the area of the base and the height. Āryabhaṭa has evidently failed to realize that the areas of similar figures are proportional to the *squares* of the corresponding sides. The area of a circle is correctly given as half the circumference multiplied by the radius and the volume of a sphere as the last multiplied by its own root,* on the analogy, perhaps, of the volume of a cube which is the area of the base multiplied by its square-root. It may be remarked here that though the Greeks had obtained correct formulæ for the above, the Hindus fell into an error—a clear indication to show that the Hindus did not owe any of their mathematics to the Greeks but that they had developed their mathematics in their own way according to their peculiar needs and idiosyncrasies. As Mr. David Eugene Smith remarks elsewhere, the mathematical taste, the purpose, and the method were all distinct in the two great divisions of the world then known.

Two other mensuration formulæ given are both correct, (i) for the lengths of the segments of the diagonals of a trapezium, (ii) for the area of a trapezium. They indicate that the Hindus must have been acquainted with the fundamental property of similar triangles. The property must have been perceived as an axiom more or less intuitively.† The Hindus were specially interested in the isosceles trapezium which was the shape of the वेदी at the Soma sacrifices discussed in the Sulva sutras.

* Sudhākara Dvivedi attempts to give a plausible explanation of Āryabhaṭa's formula for the volume of a sphere by neglecting a fraction as great as 11/32 (*vide* his Preface to the Mahāsiddhānta, p. 23). There is no evidence to show that the early Hindus neglected to take into account fractions so big as $\frac{1}{3}$.

† In the Geometry of the early Hindus, there is no theory of parallels but there is ample evidence to show that they had, instead, a theory of similar triangles. I believe that they must

There is, next, a general direction for determining the area of any figure by decomposing it into trapezia. This is just similar to the method used in modern field-surveying. One particular case of inscribing a regular polygon, *i.e.*, a hexagon within a circle is suggested by the result that the side of the regular hexagon is equal to the semi-diameter.

The Value of II and the Sine-Table.

The value of II is given by the following proportion :—

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणां ।
अयुतद्वय विष्कम्भस्यासन्नो वृत्तपरिणाहः ॥ Sloka 10.

When the diameter is 20,000, the circumference will be 62,832 approximately.

The critics say that it seems doubtful how far the above accurate value of II was used. In fact it is remarkable that Āryabhaṭa should have given it, when nothing like it occurs in the Greek works. But the fact that it is given by Āryabhaṭa immediately before his rule for the formation of the sine-tables leads one naturally to suppose that the above value of II was used only for the construction of the sine-tables at intervals of $3\frac{3}{4}°$ and that the less approximate values such as $\sqrt{10}$ were used elsewhere. This is, of course, proper.

In the Sūryasiddhānta, edited by E. Burgess and G. Whitney (p. 200), they observe that before the Greeks used the sines in calculations for the chords, they had been long employed by the Hindus. What is remarkable is the Indian invention of the semi-chord or अर्धज्या, while, as remarked by Delambre, Ptolemy himself, who came so near it, should have failed of it.

Āryabhaṭa gives the following rule for deriving the successive sine-differences. It corresponds to the well-known differential formula

$$\frac{d^2\ (\text{Sin}\ x)}{dx^2} = -\text{Sin}\ x$$

प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धं ।
तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि ॥ *

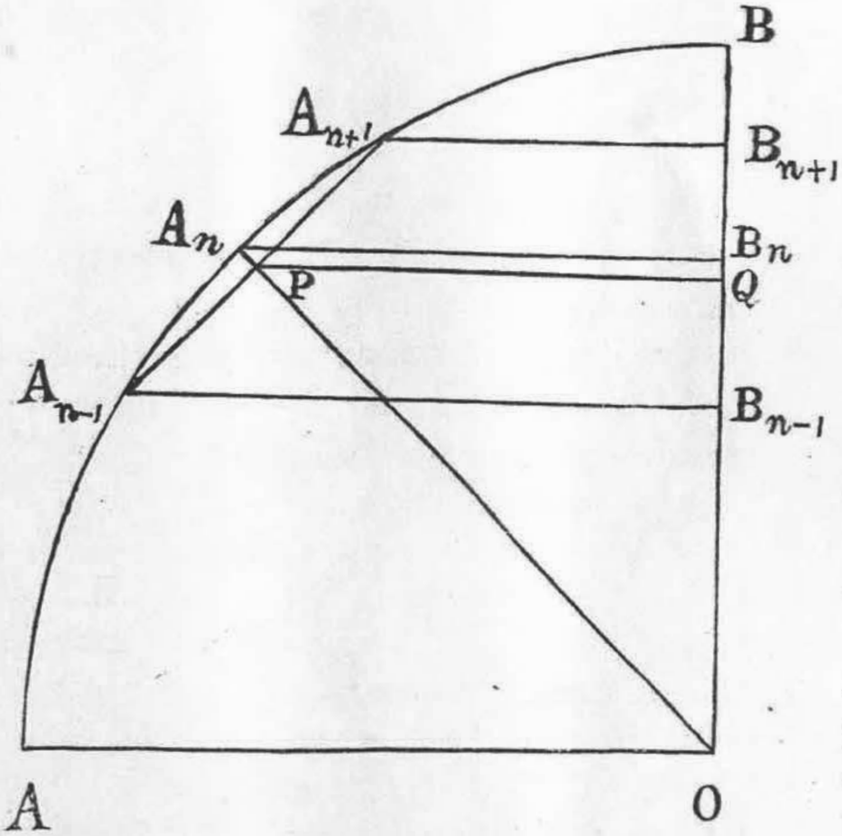
The term 'Sine' is equivalent to the modern sine multiplied by the radius 3438. According to the rule, each sine-difference diminished by the quotients of all the previous differences and itself by the first difference (*viz.*, 225),

have intuitively perceived the truth of the postulate, *viz.*, two intersecting straight lines cutting two parallel straight lines form triangles whose corresponding sides are proportional. This axiom is at the back of all their geometrical theorems, especially the well-known property of the right-angled triangle attributed to Pythagoras. That there is a remarkable anticipation of modern ideas in such an axiom as the above will be appreciated by the reader who is acquainted with the present movement in the Teaching of Elementary Geometry, to replace Euclid's parallel postulate by the postulate of similarity due to Wallis. (*Vide* The Mathematical Gazette, London, Vol. XI, p. 413, Vol. XII, p. 167, and p. 191.)

* For a complete discussion of this rule, *vide* the author's 'The Hindu Sine-Table' in J.I.M.S., Vol. XV, pp. 121-126.

gives the next difference. The differences as given in Daśagītikā are: 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, 7. The same results are also given in the Sūryasiddhānta with the same rule for obtaining them. It is significant that this rule is not quoted by Bhāskara and others. Apparently, they did not grasp its true import and ignored it.

In J.A.S.B., Vol. IV, No. 3 (pp. 123-125), Mr. G. R. Kaye holds that the above rule may be a rough attempt at the enunciation or application of Ptolemy's Theorem. But the Trigonometry of Ptolemy does not give it and indeed, as Delambre says, in order to find some vestige of it, one must, after having vainly pored over all the authors on Trigonometry, come to Briggs (1561-1631) who knew that divisor 225. Burgess and Whitney in their edition of the Sūryasiddhānta suggest that the rule may have been arrived at empirically.* But even this is not likely, as it is difficult to pitch upon the right divisor purely by guessing. The early Hindus might have obtained the result by some such reasoning as the following:—



If the radii OA, OB and the arc AB bound the quadrant AOB, and the quarter-circumference AB be divided into twenty-four equal parts so that each part is $3\frac{3}{4}°$ and perpendiculars drawn from the points of division on OB, these perpendiculars intercept on OB segments corresponding to the successive sine-differences. In particular, let A_{n-1}, A_n, A_{n+1} be three consecutive points of division on the arc AB and B_{n-1}, B_n, B_{n+1} the corresponding feet of the perpendiculars from A_{n-1}, A_n, A_{n+1} on OB; let OA_n cut $A_{n-1}A_{n+1}$ at its middle point P and let Q be the foot of the perpendicular from P on OB. (*Vide* Figure.)

* On pp. 107, 108 of J.I.M.S., Vol. XV, No. 7, February 1924, Mr. Naraharayya explains elaborately the Sūryasiddhānta rule for the calculation of successive sine-differences, echoing the views of Delambre and Rev. Burgess that the Hindus may have obtained the rule by the observation of the series 1, 2, 3. . . etc. In explaining a proof of the formula, he makes a free and

Now, it is easily seen from the property of the trapezium A_{n+1} B_{n+1} B_{n-1} A_{n-1} and the similar triangles that

$$B_{n-1}B_n - B_n B_{n+1} = 2\, QB_n$$

$$= \frac{2OB_n}{OA_n} \cdot PA_n$$

$$= \frac{OB_n . A_n A_{n+1}^2}{OA_n}$$

$$= \frac{OB_n.}{225} .$$

$$\text{for } \frac{OA_n}{A_n A_{n+1}} = \frac{\text{radius}}{\frac{1}{96} \text{ circumference}} \text{ approximately}$$

$$= \frac{960000}{62832} \quad \text{(using Āryabhaṭa's values)}$$

$$= 15 \text{ to the nearest integer.}$$

Hence the rule given by Āryabhaṭa.

Mr. Kaye remarks : " Using the formula given by Āryabhaṭa and the author of the Sūryasiddhānta, we find that only five of the sines following the first can be obtained by its means and that with the seventh sine begins a discordance * between the table and the result of calculation by the rule, which finally amounts to as much as seventy minutes. If follows, therefore, that either the rule was used but corrections were made by the aid of other tables, or the table was copied wholesale ." The last part of the above statement seems to be untrue. Ptolemy's table proceeds by half degrees and his radius is 60° and to convert Ptolemy's table to the present one, one should use the change-ratio $\frac{3}{211}$. Instead of taking all this trouble, the early Hindus would sooner and more easily have derived their tables by their rule, by applying corrections when the results disagreed with the values obtained by direct calculation. Direct calculation of the sine for the common angles 30°, 45°, 60°, 75°, and 90° gives respectively 1719, 2431, †2977, 3321 and 3438.

unlicensed use of infinitesimals, tangents, and parallels, which are quite alien to the minds of the early Hindus. In Brahmagupta's and Bhāskara's texts, the calculations proceed through half angles and their complements. Thus from sine 60°, they obtain successively sine 30°, sine 15° sine $7\frac{1}{2}$°, sine $3\frac{3}{4}$° and hence the sines of the complements of 15°, $7\frac{1}{2}$°, and $3\frac{3}{4}$° and so on. There is no evidence, as Mr. Naraharayya imagines, to show that the Hindus calculated successively the sines of $3\frac{3}{4}$°, $7\frac{1}{2}$°, $11\frac{1}{4}$°, etc., by a method however remotely resembling the method suggested by him.

* To explain the apparent discrepancies after the sixth step, Mr. Naraharayya in his Note on the Hindu Sine-Table (P. 111, J.I.M.S., Vol. XV, No. 7) quotes the rule given by the commentator Ranganatha, and himself invents other rules which are quite numerous and arbitrary and not based on rational grounds.

† The Āryabhaṭa table gives 2978, while Bhāskara who says that his table is derived from Āryabhaṭa's work and the Sūryasiddhānta give the more correct value 2977.

By following the usual rule, one meets the first discrepancy at the 7th difference. But the fractional parts neglected in the 3rd, 4th, 5th and 6th differences have accumulated sufficiently to affect the 7th difference and we are therefore justified in including them in the 7th difference, which thus becomes 205 to the nearest integer, *i.e.*, Āryabhaṭa's value. Continuing the calculations further, we find the 8th difference to be 198 to the nearest integer; but sine 30° = the sum of the 8 differences = 1719, the 8th difference has therefore probably been corrected to 199. Similar corrections applied at the angles 45°, 60°, 75° and 90° to the results obtained by the usual rule give the figures of Āryabhaṭa.

Thus one can forcibly conclude that the rule was generally used but corrections were made not by the aid of other Tables as Ptolemy's (as Mr. Kaye suggests) but by comparison of the results with the actual ones obtained by direct calculation for the common angles 30°, 45°, 60°, 75° and 90°.

[In this connection, it is interesting to note that the sines of the angles between 60° and 90° can be deduced very simply by adding the sines of two suitable angles less than 60°, for example,

$$\text{sine } 71°.\,15' = \text{sine } 11°.15' + \text{sine } 48°.\,45' = 671 + 2585$$
$$= 3256;$$
$$\text{sine } 82°.\,30' = \text{sine } 22°.\,30' + \text{sine } 37°.\,30'$$
$$= 3408$$ which disagrees with Āryabhaṭa's result 3409.

This is based on the fact that the sine of any angle $x°$ between 60° and 90° can be obtained by adding the sines of the sum and the difference of 30° and the complement of $x°$.

An interesting property of the sine-differences which follows as a corollary from the above may also be noted here, *viz.*, the n difference is equal to the $(n-16)^{th}$ difference--the $(33-n)^{th}$ difference $(n>16)$. Thus the 19th difference = the 3rd difference—the 14th difference, *i.e.*, 79 = 222—143. By means of this, one can write out the last eight differences from the first sixteen.

The matter may be presented in another form also: Write the first eight differences in one row, the next eight in a second row (in the reverse order beneath the first row), and the third eight in the third row beneath the second thus

225,	224,	222,	219,	215,	210,	205,	199
119,	131,	143,	154,	164,	174,	183,	191
106,	93,	79,	65,	51,	<u>37,</u>	22,	<u>7</u>

Then one easily sees that the figures in the third row with the exception of the underlined two figures are obtained by subtracting the figures in the second row from the corresponding figures of the first row placed vertically above them.

A similar method may also be given for the calculation of the sines of angles between 60° and 90° from those of angles less than 60°.]

The Sun-Dial and Shadow-Problems.

The next mathematical topic discussed is the mathematics of the Sun-dial and the Shadows. As a preliminary to this, constructions are given for drawing a circle, a triangle (probably an equilateral triangle) given a side, and a rectangle (probably a square) given a diagonal. Directions are also given for determining experimentally the horizontal and the vertical planes by means of water and the plumb-line respectively. The ordinary Pythagorean rule is given for finding the radius of the gnomon-circle given the height of the gnomon and the length of the shadow. Then follow two rules for determining (i) the lengths of the shadow of a gnomon of given height, and (ii) the height of the source of light and its distance from two equal gnomons casting known shadows. The formulæ are as follows:

(1) When only one gnomon is considered,

$$\text{Shadow} = \frac{\text{height of the gnomon} \times \text{the distance of the light from the gnomon}}{\text{the difference between the heights (of the gnomon and the light)}}.$$

(2) When two equal gnomons are considered, the distance between the end of a shadow and the base of the light is equal to

$$\frac{\text{the length of the shadow} \times \text{the distance between the ends of the shadows}}{\text{the difference}}.$$

There is an element of ambiguity in formula (2) with respect to the denominator, *viz.*, 'difference'. It is not clear which 'difference' is meant. The text says: छायागुणितं छायाग्रविवरमूनेन भाजिता कोटी । Now छायाग्रविवरं may mean either (A) the distance between the ends of the shadows or (B) the difference between the distances of the shadow-ends from the base of the light; and ऊन may mean either (A′) the difference of the distance between the ends of the shadows and that between the gnomons, or (B′) the difference between the lengths of the shadows.

If we accept the interpretations (A), (A′) or (B), (B′), the formula is a perfectly general one and the light and the two gnomons need not be in the same vertical plane; but if we should accept the interpretations (A), (B′), or (A′), (B) the formula holds only in the particular case where the gnomons are in the same vertical plane with the light. The second or the particular case is the interpretation of the commentators, but I am inclined to hold the first view following the interpretations (B), (B′).

It is significant to note that Bhāskara gives both the above rules of Āryabhaṭa and that Bhāskara's text also favours my interpretation (*vide* Līlāvatī—छायाव्यवहार, slokas 59-60). But Brahmagupta and Mahāvīrāchārya give only the first rule and Śrīdhara does not mention either of the rules.

Mr. Kaye observes in his Notes on Indian Mathematics (J.A.S.B., Vol. IV, No. 3, p. 128) that the Hindus were at least acquainted with the inclined gnomon and quotes Pancha Siddhantika XIII, 31 (wrongly referred to as XII, 31). But there is evidence enough in Mahāvīrāchārya's Gaṇitasāra Saṅgraha to show that the Hindus had more than a casual acquaintance with the inclined gnomon. We have in the Gaṇitasāra Saṅgraha (Text, p. 156)

स्तम्भस्य अवनतिसंख्यानयन सूत्रम् —
छायावर्गाच्छोद्या नरभाकृति गुणित शङ्कुकृतिः ।
सैकनरच्छायाकृतिगुणिता छायाकृतेः शोध्या ॥
तन्मूलं छायायां शोध्थं नरभानवर्गरूपेण ।
भागं हृत्वा लब्धं स्तम्भस्यावनतिरेव स्यात् ॥

Here the inclination of the pillar is measured by the perpendicular distance of the top of the pillar from the vertical through its foot. If we denote this inclination by x, the length of the pillar by l, the length of the shadow by s, Mahāvīra's formula is

$$x = \frac{s - \sqrt{s^2 - (s^2 - l^2 r^2)(r^2 + 1)}}{r^2 + 1}$$

where r is the ratio of human shadow to human height.

An Eclipse-Problem.

A property of the circle is then enunciated, *viz*., that in a circle, the product of the arrows is the square of the semi-chord of the arc. Immediately there follows a theorem derived from this property, which is made use of in the calculation of eclipses.

If two circles cut each other at A and B and their line of centres cut the circles at C, D, E, F in order from left to right, and the common chord in X, the greatest breadth of the common portion of the two circles is called ग्रास or bite and the measure of its segments is given by the formulæ :

$$DX = \frac{DE.\ CD}{CD+EF};\quad EX = \frac{DE.\ EF}{CD+EF}$$

This easily follows from the fact that $CX.\ XE = AX^2 = DX.\ XF$ and hence

$$\frac{DX}{EX} = \frac{CX}{FX} = \frac{DX+CX}{EX+FX} = \frac{DC}{EF}.$$

In connection with the above property of the circle, Mr. G. R. Kaye observing that M. ibn Musa (820 A.D.) gives a similar result along with a formula (not given by any Hindu writer before his time) for the computation of the area of a segment of a circle, concludes that all these rules are taken from the same source, not at any rate Indian. Mr. Kaye cannot imagine that it is possible for M. ibn Musa to gather together in one place results gathered from different sources.

The Arithmetical Progression and Allied Series.

The next topic in the Gaṇitapāda is the arithmetical progression. The following general formula is given for the sum of the terms of an A. P. beginning with the $(p+1)^{th}$ term. $n\left\{a + (\frac{n-1}{2} + p)d\right\}$,

a being the मुख or the first term and d being the चय or the increment. This formula is of special significance as we shall see later on. An alternative form of this result is suggested also :

Add the beginning and the end terms and multiply the sum by half the number of terms.

This rule is quite correct but Mr. Kaye misunderstands it (assuming आदि to mean मुख) and condemns it as out of place.

The above rule is followed by another for determining the number of terms in an A. P. given the other usual data. In the usual notation, $n = \frac{1}{2}\left\{\frac{\sqrt{8ds + (2a-d)^2} - 2a}{d} + 1\right\}$. The same rules are given, though with slight variations in form, by Brahmagupta, Bhāskara, Śrīdhara and Mahāvīrāchārya. I may add *én passant* that Mahāvīrāchārya's व्युत्कलित operation (Translation, Gaṇitasāra Saṅgraha, pp. 34-36) just corresponds to the first formula $n\left\{a + (\frac{n-1}{2} + p)\, d\right\}$ and this formula occurs neither in Brahmagupta's nor in Bhāskara's works.

Āryabhaṭa then proceeds to give the contents of a triangular pile and a square pile as $\frac{n(n+1)(n+2)}{6}$ or $\frac{(n+1)^3-(n+1)}{6}$ and $\frac{n(n+1)(2n+1)}{6}$ respectively. The formula for the sum of the cubes is given by

$$1^3 + 2^3 + 3^3 + .. + n^3 = (1 + 2 + 3 + .. + n)^2$$

The latter formula might have been derived from the series (1), (3, 5), (7, 9, 11), (13, 15, 17, 19), etc., divided into groups as shown by the brackets. The sum of the numbers in each group is a perfect cube as can be seen on the application of Āryabhaṭa's first formula. Hence the given result follows easily by expressing the sum of the n groups as the sum of $(1+2+3+ .. +n)$ natural odd numbers.

Brahmagupta and Bhāskara also give the above formula, but Mahāvīrāchārya advances very much beyond these writers, for he gives expressions for (1) the sums of the squares of the terms of an A.P., (2) the sums of the cubes of the terms of an A.P., and (3) the sum of a series wherein each term represents the sum of a series of natural numbers up to a limiting number which is itself a member in a series in arithmetical progression, etc. *Vide* his Gaṇitasāra Saṅgraha, pp. 169-173 (Translation by M. Rangacharya).

On closer examination of the form of Mahāvīrāchārya's results, it will suggest to one that his व्युत्कलित operation corresponding to Āryabhaṭa's first formula was the key to obtain the values of Σn^2, Σn^3, $\Sigma n(n+1)$, etc. For example, take the series of natural numbers 1, 2, 3, ... and group them as follows :—

$$(1+2+3)+(4+5+6+7+8)+(9+10+11+12+13+14+15)+\ldots.$$

the first group containing 3 terms, the second 5, the third 7 and so on. The number of terms preceding the n^{th} group is $3+5+\ldots+\overline{2n-1}$, *i.e.*, n^2-1 and the n^{th} group contains $2n+1$ terms. So, by Āryabhaṭa's formula, the sum of the terms in the n^{th} group is

$$(2n+1)\left(1+\frac{2n+1-1}{2}+n^2-1\right),\ i.e.,\ n(n+1)(2n+1)$$

Now, on comparing the series formed by the sums of the groups with the series 1^2, 1^2+2^2, $1^2+2^2+3^2$, $1^2+2^2+3^2+4^2, \ldots$ one finds that the sum of the terms in any group is 6 times the corresponding term in the above series.

$$\text{Hence } 6(1^2+2^2+\ldots+n^2)=n(n+1)(2n+1)$$

$$i.e.,\ 1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.\text{*}$$

Similarly, by grouping the series of natural numbers in another way: $\overline{1+2}$, $\overline{3+4+5}$, $\overline{6+7+8+9}, \ldots$ etc., and comparing the series formed by the sums of the groups with the series 1.2, 1.2+2.3, 1.2+2.3+3.4, ... we find that the ratio of the corresponding terms in the two series is 3/2. But the sum of the terms in the n^{th} group of the first series is, according to Āryabhaṭa's rule, obtained as $\frac{n(n+1)(n+2)}{2}$. Hence the n^{th} term of the second series, *viz.*, $1.2+2.3+\ldots$ to n terms $=\frac{2}{3}\cdot n\frac{(n+1)(n+2)}{2}$ which leads to the required result.

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* It is not unlikely as pointed out by Dr. R. P. Paranjpye that the sum of the squares of the first n natural numbers may have been discovered by the early mathematicians by arranging the numbers 1, 2, 3, . . . each repeated $2n+1$ times in a rectangular array and grouping the elements together as indicated below :

$$\begin{array}{ccc|cc|cc|l}
1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots\ (2n+1)\text{ terms} \\
\hline
2 & 2 & 2 & 2 & 2 & 2 & 2 & \ldots \\
\hline
3 & 3 & 3 & 3 & 3 & 3 & 3 & \ldots \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
n & n & n & n & \ldots & & &
\end{array}$$

The total sum of all the numbers $=\frac{n(n+1)(2n+1)}{2}$ and the sums of the numbers in the gnomons are, in order, $3, 3.2^2, 3.3^2, \ldots$ Hence $3(1^2+2^2+3^2+\ldots)=\frac{n(n+1)(2n+1)}{2}$ which leads to the formula in question.

Similarly other cases of summation given in Gaṇitasāra Saṅgraha can be worked out. The main difficulty is to get at a suitable arithmetical progression and to group the terms appropriately. [I have discovered a rule for finding such a progression (*vide* my note on ' Series Summable as Arithmetical Progressions' in Mathematical Notes No. 23, June 1925, published by the Edinburgh Mathematical Society). Suppose it is required to find $\Sigma(An^3+3Bn^2+Cn+D)$. In the first place, it is possible to find a suitable A. P. only when $2B^3 = A(BC-AD)$ and A, $A+B$ are of the same sign, and $A \pm 0$; if these conditions be satisfied, the first term and the common difference may be taken respectively as $\frac{1}{2}\left(\frac{B^2+AD}{Bt}+\frac{A}{t^2}\right)$ and $\frac{A}{t^2}$, t being an arbitrary positive integer such that $\frac{t(B+A)}{A}$ is also a positive integer. The series should be grouped such that the first group may contain $\frac{t(B+A)}{A}$ terms, the second group $\frac{t(B+2A)}{A}$ terms, the third group $\frac{t(B+3A)}{A}$ terms, and so on.]

Some Semi-Geometrical Identities.

The next topic in Āryabhāṭa's mathematics is a pair of semi-geometrical identities, *viz.*,

$$(a+b)^2-(a^2+b^2)=2ab\,;\quad \sqrt{4ab+(a-b)^2} \pm (a-b) = 2a \text{ or } 2b.$$

It is likely that these identities were intended to solve simultaneous equations of the types :

$$x \pm y=p,\ xy=q\,;\ x \pm y=p,\ x^2+y^2=q\,;\ xy=p,\ x^2+y^2=q.$$

Interest Problems.

Āryabhaṭa then gives a rule for finding the interest on the Principal, given the principal (P), the interest on the amount (A), and the time (t). The rule is obtained from the quadratic equation *

$$Pr^2t^2+Prt=A.$$

The problem suggests that even in the olden days money-lenders were deducting Banker's interest at the outset while lending money. As Mr. Kaye says, considerable acquaintance with the rules that govern interest problems must have obtained in those times and at least the rudiments of compound interest were understood. For various types of interest problems, there is not a wealthier storehouse than Mahāvīrāchārya's Gaṇitasāra Saṅgraha,

* Here, as well as in the Section on Series, Āryabhaṭa casually introduces the general formula for the solution of the quadratic equation. Possibly the problem of attacking the quadratic equation must have been faced by the early mathematicians for the first time in connection with the inverse problem of finding the number of terms of an A. P. or the rate per cent in a compound interest calculation.

(Text, pp. 65-75) वृद्धि विधान where he treats of the interest problems as illustrative of the principle of the Double Rule of Three or पञ्चराशिक.

The Rule of Three and Operations with Fractions.

After giving the rule for finding interest, Āryabhaṭa proceeds to enunciate the principle of the rule-of-three or त्रैराशिक and the usual rules for the division of one fraction by another and for reducing all fractions to a common denominator. Bhāskara, Brahmagupta, Śrīdhara and others use the same nomenclature and extend the rule of three to five, seven, nine, and eleven terms.

The Rule of Inverse Operations.

The rules for reversing the steps in a mathematical process (called व्यस्तविधि) are enunciated thus by Āryabhaṭa :—

गुणकारा भागहरा भागहरा ये भवन्ति गुणकाराः ।
यः क्षेपस्सोपचयोऽपचयः क्षेपश्च विपरीते ॥

Every operation in algebra is connected with another which is exactly opposite to it in effect, *i.e.*, what is done by one is undone by the other; thus we have the pairs of inverse operations: addition and subtraction; multiplication and division; root and power, etc. This principle is very useful for verification purposes and also for the solution of equations where one has to clear the variable from all the ramifications in which it is involved. It is specially serviceable in solving the so-called 'think of a number' problems or boomerang problems. For example, Āryabhaṭa's commentator, Paramādīśwara, gives this illustration:

What is the number which, multiplied by 3, divided by 5, the quotient increased by 6, the square-root of the sum diminished by 1, and again squared yields the result 4?

The result obtained by reversing the operations in order may be expressed as $\left\{(\sqrt{4}+1)^2-6\right\} 5 \div 3$, *i.e.*, 5. Bhāskara and others give similar rules.

An Algebraic Identity.

We next come to a very elegant identity which is wrongly believed to be a plagiarism from Greek source. It is in Āryabhaṭa's words:

राश्यूनं राश्यूनं गच्छधनं पिण्डितं पृथक्त्वेन ।
व्येकेन पदेन हृतं सर्वधनं तद्भवत्येव ॥

In modern notation, $\sum_{1}^{n} \left(\sum X_r - X_r\right) \div (n-1) = \sum_{1}^{n} X_r$. The rule is quite simple and not beyond Āryabhaṭa's mathematics. From a fancied resemblance to the Greek Theorem known as the *Epanthem* of Thymaridas

(Diophantus, p. 115) and from the fact that two particular cases of this theorem occur in Diophantus (p. 135), it is argued by Cantor and Mr. Kaye that the problem is of Greek origin. In this connection, it may be well to point out that Mahāvīrāchārya gives just the problem which, Iamblichus says, can be reduced to Thymaridas' form. [*Vide* Diophantus, pp. 115, 116 and Gaṇitasāra Saṅgraha, pp. 95 (Text), 153 (Translation by M. Rangacharya).] But Mahāvīrāchārya (*Ibid.*, pp. 153-163, Translation by M. Rangacharya) gives many other plentiful varieties of problems not found in Diophantus, Book I or in any other ancient Greek work.

As Mr. E. B. Havell remarks in his 'History of Aryan Rule in India' (pp. 140-141), it would be wrong to conclude that the mercantile relations between Greece and India had any deep or abiding influence upon Indian culture or upon the religious movements of the times. Hellenistic culture drew more inspiration from Indian influence than Indian culture from Hellenistic influence. India always gave others more than she took from them. Possibly the seeds of Indian Arithmetic and Algebra which flowered later in the Alexandrian School were laid there by the early Dravidian traders who carried the natural products of South India to Babylon, Egypt and Greece. Probably also the Sumerian founders of Babylonia were of Dravidian stock as the striking resemblances in ethnic type would show. (*Vide* Hall's 'History of the Near East', pp. 173-174.)

The comparatively greater perfection of Greek Mathematics as embodied in such works as those of Euclid, Ptolemy, and Diophantus and in contrast with it the elementary and fragmentary nature of Indian mathematics with, of course, exceptionally brilliant development in certain directions, are proof sufficient to conclude that the latter is of indigenous growth and not borrowed from the former. It is not proper to argue that Indians had stolen from the Greek works merely on the score that one or two of their problems had been anticipated in the early Greek or Alexandrian works. One may also add that the ancient Indians lacked the will, if not the genius, which the Arabs possessed in a high degree, to translate foreign works into their own language.

The Simple Equation and Relative Velocity.

After the so-called Indian version of the *Epanthem* follows the ordinary method for the solution of a simple equation where both the sides are linear functions of the variable. This is succeeded by a discussion of the relative velocity of one moving body with respect to another, when both are moving (i) in the same direction, and (ii) in opposite directions.

The Linear Indeterminate Equation.

We now come to the very crown of Āryabhaṭa's mathematics—his solution of the linear indeterminate equation. He puts the problem thus:

* To find a number which leaves residues n, n' with respect to the moduli m, m' respectively.

If $n > n'$, m is called अधिकाग्रभागहार or the divisor corresponding to the greater residue (not the greater divisor as wrongly translated by Mr. Kaye) and m' is called ऊनाग्रभागहार or the divisor belonging to the lesser residue (not necessarily the lesser divisor), while the residue for the modulus mm' is called द्विच्छेदाग्र.

To quote Āryabhaṭa's words: —

अधिकाग्रभागहारं छिन्द्यादूनाग्रभागहारेण ।
शेष परस्परभक्तं मतिगुणमग्रान्तरे क्षिप्तं ॥
अधउपरिगुणितमन्त्ययुगूनाग्रच्छेदभाजिते शेष ।
मधिकाग्रच्छेदगुणं द्विच्छेदाग्रमधिकाग्रयुतम् ॥

Divide अधिकाग्रभागहार † by the other divisor and continue the process with the remainders. Write out the successive quotients in a vertical line, one underneath the other. Choose a suitable integer (called *mati*) which when multiplied by the final remainder and added to the difference between the given residues may yield an integral quotient when divided by the final divisor. Set down the *mati* beneath the last quotient and beneath it place the aforesaid integral quotient. Multiply the lower by the upper and add the last and continue this process till the operations cannot be further pursued. Divide (if possible) the figure thus obtained by the first divisor and multiply the remainder by the second (divisor). The product added to the corresponding residue is the required result.

The rationale and the genesis of this method can best be explained by an example.

Let $m'=29$, $n'=15$, $m=45$, $n=19$

We have to find x and y to satisfy the equation :—$29x+15=45y+19$.

The method that immediately suggests itself is to express x in terms of y, *i.e.*, $x=y+\frac{16y+4}{29}$

Since $\frac{16y+4}{29}$ should be an integer, put $\frac{16y+4}{29}=z$

Then $y=z+\frac{13z-4}{16}$. Again set $\frac{13z-4}{16}=p$, so that $z=p+\frac{3p+4}{13}$. At this stage since the co-efficient of p is small, we readily choose the *mati* (*viz.*) $p=3$ which makes $3p+4$ divisible by 13.

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* The form in which Āryabhaṭa has worded this problem makes it very probable that the linear indeterminate equation was first studied in India merely as an inverse problem under division. The later Hindu mathematicians must subsequently have discovered a good application of it in verifying astronomical calculations as well as in obtaining simple approximations to unwieldy fractions frequently occurring in astronomy.

† If अधिकाग्रभागहार be the smaller of the two divisors we have only to put 0 (zero) for the first quotient and write down the number itself as the remainder, using it next as the divisor for the other हार. The division may be further continued as directed.

Now to find the value of x, we have only to retrace our steps and get $z=4$, $y=7$, $x=11$. This is just the process which Āryabhaṭa asks us to do, detaching the co-efficients of y, z and p, (*mati*), and $\frac{3p+4}{13}$. Āryabhaṭa's rule is therefore nothing more than a method of detached co-efficients for carrying out the backward process of evaluating successively z, y, and x. We may arrange the successive columns of reduction thus :

$$\begin{array}{cccc} 1 & 1 & 1 & 11 \\ 1 & 1 & 7 & 7 \\ 1 & 4 & 4 & \\ 3 & 3 & & \\ 1 & & & \end{array}$$

Now, 11 and 7, the two top figures, are the values of x and y respectively. The required number is $29\times 11+15$, *i.e.*, 334. It may also be obtained by using the other divisor 45 and the corresponding residue 19; thus $45\times 7+19$ also gives 334.

In the above method we are reducing the given indeterminate equation to others of simpler form with smaller co-efficients and we may stop our continued division at any stage where the co-efficients are sufficiently small to enable us to read the results immediately. Thus, if we can easily see at the second stage itself that z must be 4 which is मति, we may form the shorter *valli* from which we immediately derive the value 11 by the process अध उपरि गुणितं, etc.

$$\begin{array}{ll} 1 & \multirow{2}{*}{$\begin{pmatrix} 11 \\ 7 \end{pmatrix}$} \\ 1 & \\ 4 & \\ 3 & \end{array}$$

The above process is styled by later Hindu mathematicians as Vallikā-kuṭṭākāra (वल्लिकाकुट्टाकार:). In Bhāskara's method the creeper or *valli* is extended to its utmost length. (*Vide* Līlāvatī verse 67.) Thus in the previous example, we may put $\frac{3p+4}{13}=q$; then $p=4q+\frac{q-4}{3}$; and put again $\frac{q-4}{3}=r$, so that $q=3r+4$; lastly set $r=0$. So Bhāskara's *valli* will be as shown in the margin, the rest of the process being the same as before. But one important point which Brahmagupta and others give, Āryabhaṭa fails to mention, *viz.*, if the final quotient in the creeper be of the odd order (as happens in the marginal illustration) the result obtained (from the manipulation of the elements of the creeper) is positive ; otherwise, the result is negative ; and hence to derive a positive result in the latter case one has to *subtract* its numerical value from a suitable multiple of the अधिकाग्रभागहार.

$$\begin{array}{c} 1 \\ 1 \\ 1 \\ 4 \\ 3 \\ 0 \\ 4 \end{array}$$

It is interesting to note that Mahāvīrāchārya gives exactly Āryabhaṭa's rule using Āryabhaṭa's nomenclature.

भाज्यच्छेदाग्रशैषैः प्रथमहृतिफलं त्याज्यमन्योन्यभक्तं
न्यस्यान्ते साग्रमूर्ध्वैरुपरिगुणयुतं तैस्समानासमाने ।
स्वर्णघ्नं व्याप्तहारौ गुणधनमृणयोश्चाधिकाग्रस्य हारं
हृत्वा हृत्वा तु साग्रान्तरधनमधिकाग्रान्वितं हारघातम् ॥

(The nomenclature of Āryabhaṭa is underlined in the verse.)

[In this connection I may suggest an alternative method for the solution of the linear indeterminate equation, based upon the method of expressing any fraction as the sum or difference of fractions with unit numerators. This process was not unknown to the early Hindus, for we find in the Sulva Sutras (200 A.D.), $\sqrt{2}$ expressed approximately in this form, *viz.*, $1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34}$. The method may best be illustrated by an example, say, $32x+17=79y+10$.

To express $\frac{32}{79}$ as the sum or difference of unit fractions, we have only to divide 79 successively by 32 and the absolutely least residues obtained during the process :

$$\begin{array}{l}
32)\ 79\ (2 \\
\quad\ \ 64 \\
\quad\ \ \overline{15)\ 79\ (5} \\
\qquad\quad\ 75 \\
\qquad\quad\ \overline{4)\ 79\ (20} \\
\qquad\qquad\ \ 80 \\
\qquad\qquad\ \overline{-1)\ 79\ (-79} \\
\qquad\qquad\qquad\ \ 79 \\
\qquad\qquad\qquad\ \ \overline{\ 0\ }
\end{array}$$

Thus $\frac{32}{79}=\frac{1}{2}-\frac{1}{2.5}+\frac{1}{2.5.20}+\frac{1}{2.5.20.79}$

The given equation may be reduced to the form

$$\frac{32x}{79}=y-\frac{7}{79}$$

$$i.e.,\ x\ \left\{\frac{1}{2}-\frac{1}{2.5}+\frac{1}{2.5.20}+\frac{1}{2.5.20.79}\right\}=y-\frac{7}{79}$$

Obviously, if we put $x=2.5.20.(-7)$, y becomes an integer.

The least values of x and y are 22 and 9.

N.B.—1. Since we are taking the absolutely least residue at each step, the number of unit fractions cannot exceed $\log_2$ (divisor).

2. The above method will fail sometimes when the greater of the two co-efficients, those of x and y is composite ; for example, let $11x=72y+13$, which expressed in the above form leads to $x\left\{\frac{1}{6}-\frac{1}{72}\right\}=y+\frac{13}{72}$. But if we put $x=-13$, $y=-\frac{13}{6}$, a fractional value. Thus we fail to get an integral solution. This is due to the fact that one of the residues in the process of division happens to contain a factor of the dividend ; and if the co-efficient of y had been prime, such an occurrence would be impossible. Hence in such cases

we may add to the greater co-efficient (*e.g.*, 72) a suitable multiple of the other co-efficient (*viz.*, 11) such that the sum may be a prime and this is certainly possible from Dirichlet's Theorem, *viz.*, that $mz+n$ represents infinitely many primes if m and n are relatively prime. Thus since $72+11 = 83$ a prime, we can solve by the above method $11x = 83y + 13$ and get $y = 7$ which holds good for the original equation also.]

It is obvious that Āryabhaṭa's rule is more primitive than that of his successors who have elaborated upon the rule and pushed it to its logical conclusion. When analysed, the rule implies merely successive reductions to simpler indeterminate equations until one is reached whose solution can be immediately guessed. This is the true significance of *mati* in Āryabhaṭa's verse.* It is just the method which will naturally suggest itself to any gifted mind and it is therefore no wonder that Āryabhaṭa or any of his predecessors should have discovered it.

The theorem underlying the rule is not really complicated as Mr. G. R. Kaye imagines, unless one reads into it all the modern algebraic ramifications of the general continued fraction. Hence, it is not necessary to go in search of the orderly processes by which, according to our orientalist, such a complicated theorem is bound to be preceded. Mr. Kaye evidently confuses between the logical and the psychological orders of evolution of mathematical ideas. Psychologically, there is ample justification for the development of the above ideas in Āryabhaṭa's mind without a previous knowledge of such preliminary notions as set forth by Greek writers, especially Euclid. The fact that we nowhere find in the Greek works the rule as given by Āryabhaṭa or anything analogous to it (however remote the analogy as in the case of the *Epanthem*) is sufficient justification to attribute to Āryabhaṭa or the early Hindus the first foundation of Indeterminate Analysis.

Another point to note in this connection is that long before the Alexandrian Christians had begun to wrangle about the dates in the ecclesiastical calendar, the Hindus had felt an urgent need for developing their Indeterminate Analysis to help them to verify their huge astronomical calculations. For, as Bhāskara puts it,

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* Failing to understand this aspect of *mati*, one of the commentators on Āryabhaṭa, Devaraja by name, puts the following query in his 'Kuṭṭākāra Çirōmani':—

'When the problem can be solved without the trouble of choosing a *mati* as in Bhāskarā's method, why should it not have been adopted by Āryabhaṭa?'

He defends the use of *mati* on the score that it is not necessary for problems on निरग्रकुट्टाकार discussed by Bhāskara. He does not realize that both साग्रकुट्टाकार and निरग्रकुट्टाकार are practically one and the same and *mati* is no more essential in the one than in the other. Evidently the commentator does not perceive that Āryabhaṭa's method is more primitive than Bhāskara's, which marks a later stage in the evolution of the linear Indeterminate Analysis.

अस्ति अस्य ग्रहगणिते महानुपयोगः । तदर्थं किंचिदुच्यते—कल्प्याथशुद्धिर्विकलावशेषं षष्टिश्चभाज्यः कुदिनानि हारः । . . .

'There is great use for this process in mathematical astronomy in the calculation of lapsed terrestrial days from the residual seconds, etc.' (*Vide* Prasna Adhyāya, verses 11-22. Bhāskara's Gōlādhyāya.) There is thus greater reason for claiming, on behalf of the Hindus the development of the Indeterminate Analysis, than on behalf of the wrangling Christians of Alexandria.

Conclusion.

The Gaṇitapāda ends with the Indeterminate Equations. The rest of Āryabhaṭa's work is astronomy. On the whole, the impression left by the Gaṇita is that it is a collection of working rules necessary for solving the practical problems of life such as survey and interest problems and the practical problems of astronomy which are closely connected with a Hindu household even to-day. The author's style is coldly business-like, lacking the richness of imagination, the zeal in problem-setting, and the extravagant poetry characteristic of other Indian authors, for example. Bhāskara and Mahāvīrāchārya. Very likely, Āryabhaṭa's work has superseded the work of earlier Indian writers in the field and in default of discovery of fresh manuscripts in unexplored libraries, it must be idle to speculate how much of his work is really original.

Enough has been said in the previous pages to show to what extent the later Hindu astronomer-mathematicians were indebted to Āryabhaṭa. He was the first to give a form and an individuality to the scattered bits of mathematical knowledge that existed before his time and but for his pioneer work, there is no knowing what turn subsequent Indian Mathematics would have taken. The rôle of the Āryabhaṭīyam in giving a definite bias to Indian Mathematics has its historic parallel and counterpart only in two other great ancient mathematical compositions—the Elements of Euclid and the Arithmetica of Diophantus.